

# Recurrence in the dynamical system $(X, \langle T_s \rangle_{s \in S})$ and ideals of $\beta S$

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## Abstract

A *dynamical system* is a pair  $(X, \langle T_s \rangle_{s \in S})$ , where  $X$  is a compact Hausdorff space,  $S$  is a semigroup, for each  $s \in S$ ,  $T_s$  is a continuous function from  $X$  to  $X$ , and for all  $s, t \in S$ ,  $T_s \circ T_t = T_{st}$ . Given a point  $p \in \beta S$ , the Stone-Čech compactification of the discrete space  $S$ ,  $T_p : X \rightarrow X$  is defined by, for  $x \in X$ ,  $T_p(x) = p\text{-}\lim_{s \in S} T_s(x)$ . We let  $\beta S$  have the operation extending the operation of  $S$  such that  $\beta S$  is a right topological semigroup and multiplication on the left by any point of  $S$  is continuous. Given  $p, q \in \beta S$ ,  $T_p \circ T_q = T_{pq}$ , but  $T_p$  is usually not continuous. Given a dynamical system  $(X, \langle T_s \rangle_{s \in S})$ , and a point  $x \in X$ , we let  $U(x) = \{p \in \beta S : T_p(x) \text{ is uniformly recurrent}\}$ . We show that each  $U(x)$  is a left ideal of  $\beta S$  and for any semigroup we can get a dynamical system with respect to which  $K(\beta S) = \bigcap_{x \in X} U(x)$  and  $\text{cl}K(\beta S) = \bigcap \{U(x) : x \in X \text{ and } U(x) \text{ is closed}\}$ . And we show that weak cancellation assumptions guarantee that each such  $U(x)$  properly contains  $K(\beta S)$  and has  $U(x) \setminus \text{cl}K(\beta S) \neq \emptyset$ .

## 1 Introduction

We take the Stone-Čech compactification of a discrete semigroup  $(S, \cdot)$  to be the set of ultrafilters on  $S$ , identifying the points of  $S$  with the principal ultrafilters. Given  $A \subseteq S$ , we set  $\bar{A} = \{p \in \beta S : A \in p\}$ . The set  $\{\bar{A} : A \subseteq S\}$  is a basis for the open sets and a basis for the closed sets of  $\beta S$ . The operation on  $S$  extends uniquely to  $\beta S$  so that  $(\beta S, \cdot)$  is a right topological semigroup with  $S$  contained in its topological center, meaning that  $\rho_p$  is continuous for each

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$p \in \beta S$  and  $\lambda_x$  is continuous for each  $x \in S$ , where for  $q \in \beta S$ ,  $\rho_p(q) = q \cdot p$  and  $\lambda_x(q) = x \cdot q$ . So, for every  $p, q \in \beta S$ ,  $pq = \lim_{s \rightarrow p} \lim_{t \rightarrow q} st$ , where  $s$  and  $t$  denote elements of  $S$ . If  $A \subseteq S$ ,  $A \in p \cdot q$  if and only if  $\{x \in S : x^{-1}A \in q\} \in p$ , where  $x^{-1}A = \{y \in S : xy \in A\}$ . (We are following the custom of frequently writing  $xy$  for  $x \cdot y$ .)

The algebraic structure of  $\beta S$  is interesting in its own right, and has had substantial applications, especially to that part of combinatorics known as *Ramsey Theory*. See the book [4] for an elementary introduction to the structure of  $\beta S$  and its applications.

We are concerned in this paper with the relationship between the algebraic structure of  $\beta S$  and recurrence in *dynamical systems*.

**Definition 1.1.** A *dynamical system* is a pair  $(X, \langle T_s \rangle_{s \in S})$  such that

- (1)  $X$  is a compact Hausdorff topological space (called the *phase space* of the system);
- (2)  $S$  is a semigroup;
- (3) for each  $s \in S$ ,  $T_s$  is a continuous function from  $X$  to  $X$ ; and
- (4) for all  $s, t \in S$ ,  $T_s \circ T_t = T_{st}$ .

Associated with any semigroup  $S$  are at least two interesting dynamical systems, namely  $(\beta S, \langle \lambda_s \rangle_{s \in S})$ , and  $(^S\{0, 1\}, \langle T_s \rangle_{s \in S})$  where  $^S\{0, 1\}$  is the set of all functions from  $S$  to  $\{0, 1\}$  with the product topology and  $T_s(x) = x \circ \rho_s$ . (We shall verify that this latter example is a dynamical system shortly.)

It is common to assume that the phase space of a dynamical system is a metric space, but we make no such assumption. If  $S$  is infinite, then  $\beta S$  is not a metric space. Everything we do here is boring if  $S$  is finite so whenever we write “let  $S$  be a semigroup” we shall assume that  $S$  is infinite. The interested reader can amuse herself by determining which of our results remain valid if that assumption is dropped.

The system  $(\beta S, \langle \lambda_s \rangle_{s \in S})$  has significant general properties as can be seen in [4, Section 19.1], but will not be used much in this paper.

Given a product space  $^S\{0, 1\}$ , recall that the product topology has a subbasis consisting of sets of the form  $\pi_t^{-1}[\{a\}]$  for  $t \in S$  and  $a \in \{0, 1\}$ , where, for  $x \in ^S\{0, 1\}$ ,  $\pi_t(x) = x(t)$ .

**Lemma 1.2.** *Let  $R$  be a semigroup and let  $S$  be a subsemigroup of  $R$ . Let  $Z = ^R\{0, 1\}$ , the set of all functions from  $R$  to  $\{0, 1\}$  with the product topology. For  $x \in Z$  and  $s \in S$ , define  $T_s(x) = x \circ \rho_s$ . Then  $(Z, \langle T_s \rangle_{s \in S})$  is a dynamical system.*

*Proof.* It is routine to verify that for  $s, t \in S$ ,  $T_s \circ T_t = T_{st}$ . To see that  $T_s$  is continuous for each  $s \in S$ , let  $s \in S$  be given. It suffices to show that the inverse image of each subbasic open set is open, so let  $t \in R$  and  $a \in \{0, 1\}$  be given. Then  $T_s^{-1}[\pi_t^{-1}[\{a\}]] = \pi_{ts}^{-1}[\{a\}]$ .  $\square$

Recall that, if  $T$  is any discrete space,  $p \in \beta T$ ,  $\langle x_t \rangle_{t \in T}$  is any indexed family in a Hausdorff topological space  $X$ , and  $y \in X$ , then  $p\text{-}\lim_{t \in T} x_t = y$  if and only if for every neighborhood  $U$  of  $y$ ,  $\{t \in T : x_t \in U\} \in p$ . In compact spaces  $p$ -limits always exist.

**Definition 1.3.** Let  $(X, \langle T_s \rangle_{s \in S})$  be a dynamical system and let  $p \in \beta S$ . Then  $T_p : X \rightarrow X$  is defined by, for  $x \in X$ ,  $T_p(x) = p\text{-}\lim_{s \in S} T_s(x)$ . So  $T_p(x) = \lim_{s \rightarrow p} T_s(x)$  where  $s$  denotes an element of  $S$ .

Using [4, Theorem 4.5] one easily sees that for  $p, q \in \beta S$ ,  $T_p \circ T_q = T_{pq}$ . However,  $(X, \langle T_s \rangle_{s \in \beta S})$  is not in general a dynamical system, since  $T_p$  is not likely to be continuous when  $p \in \beta S \setminus S$ . However, for each  $x \in X$ , the map  $p \mapsto T_p(x) : \beta S \rightarrow X$  is continuous. To see this, define  $f_x(p) = T_p(x)$ . If  $U$  is a neighborhood of  $f_x(p)$  and  $A = \{s \in S : T_s(x) \in U\}$ , then  $U \in p$  and  $f_x[\overline{A}] \subseteq U$ . Alternatively, one may note that  $p \mapsto T_p(x)$  is the continuous extension to  $\beta S$  of the function  $s \mapsto T_s(x) : S \rightarrow X$ .

As a compact Hausdorff right topological semigroup,  $\beta S$  has a number of important algebraic properties, and we list some of those that we shall use. (Proofs can be found in [4, Chapters 1 and 2]. Assume that  $T$  is a compact Hausdorff right topological semigroup. A non-empty subset  $V$  of  $T$  is a *left ideal* if  $tV \subseteq V$  for every  $t \in T$ , a *right ideal* if  $Vt \subseteq V$  for every  $t \in T$ , and an *ideal* if it is both a left and a right ideal.

- (1)  $T$  contains an idempotent.
- (2)  $T$  has a smallest ideal  $K(T)$ , which is the union of the minimal left ideals of  $T$  and the union of the minimal right ideals of  $T$ .
- (3) For every  $t \in K(T)$ ,  $Tt$  is a minimal left ideal of  $T$  and  $tT$  is a minimal right ideal of  $T$ .
- (4) The intersection of any minimal left ideal and any minimal right ideal of  $T$  is a group.
- (5) Every left ideal of  $T$  contains a minimal left ideal, and every right ideal of  $T$  contains a minimal right ideal.
- (5) Every minimal left ideal of  $T$  is compact.
- (6) If  $\{t \in T : \lambda_t \text{ is continuous}\}$  is dense in  $T$ , then the closure of every ideal in  $T$  is also an ideal.

We introduce the main objects of study in this paper now. Given a set  $X$ , we let  $\mathcal{P}_f(X)$  be the set of finite nonempty subsets of  $X$ .

**Definition 1.4.** Let  $S$  be a semigroup and let  $A \subseteq S$ . We say the set  $A$  is *syndetic* if and only if there exists  $F \in \mathcal{P}_f(S)$  such that  $S = \bigcup_{t \in F} t^{-1}A$ .

In the semigroup  $(\mathbb{N}, +)$  a set is syndetic if and only if it has bounded gaps.

**Definition 1.5.** Let  $(X, \langle T_s \rangle_{s \in S})$  be a dynamical system and let  $x \in X$ .

- (a) The point  $x$  is *uniformly recurrent* if and only if for every neighborhood  $V$  of  $x$ ,  $\{s \in S : T_s(x) \in V\}$  is syndetic.
- (b)  $U(x) = U_X(x) = \{p \in \beta S : T_p(x) \text{ is uniformly recurrent}\}$ .

In Section 2 of this paper we present well known results about  $U(x)$  that are valid in arbitrary dynamical systems as well as the few simple results that we have in the dynamical system  $(\beta S, \langle \lambda_s \rangle_{s \in S})$ .

In Section 3 we present results about the dynamical systems described in Lemma 1.2.

In Section 4 we consider the effect of slightly modifying the phase space in the dynamical systems described in Lemma 1.2.

In Section 5 we consider surjectivity of  $T_p$  and the set  $NS = NS_X = \{p \in \beta S : T_p : X \rightarrow X \text{ is not surjective}\}$  which is a right ideal of  $\beta S$  whenever it is nonempty.

## 2 General results

We begin with some well known basic facts.

**Lemma 2.1.** *Let  $(X, \langle T_s \rangle_{s \in S})$  be a dynamical system, let  $L$  be a minimal left ideal of  $\beta S$ , and let  $x \in X$ . The following are equivalent:*

- (a)  $x$  is uniformly recurrent.
- (b) There exists  $q \in L$  such that  $T_q(x) = x$ .
- (c) There exists an idempotent  $q \in L$  such that  $T_q(x) = x$ .
- (d) There exists  $y \in X$  and  $q \in L$  such that  $T_q(y) = x$ .
- (e) There exists  $q \in K(\beta S)$  such that  $T_q(x) = x$ .
- (f) There exists  $y \in X$  and  $q \in K(\beta S)$  such that  $T_q(y) = x$ .

*Proof.* The equivalence of (a)-(d) is shown in [4, Theorem 19.23]. Since (c) implies (e), and (e) implies (f), we shall show (f) implies (c) and this will establish the equivalence of all six statements. So assume that (f) holds. Let  $u$  denote the identity of the group  $L \cap q\beta S$ . Since  $uq = q$ , it follows that  $T_u(x) = T_u T_q(y) = T_q(y) = x$ .  $\square$

**Corollary 2.2.** *Let  $(X, \langle T_s \rangle_{s \in S})$  be a dynamical system and let  $x \in X$ .*

- (1) *If  $x$  is uniformly recurrent,  $U(x) = \beta S$ .*
- (2) *For each  $x \in X$ ,  $U(x)$  is a left ideal of  $\beta S$ .*
- (3) *For every  $x \in X$ ,  $K(\beta S) \subseteq U(x)$ .*
- (4)  *$\bigcap_{x \in X} U(x)$  is a two sided ideal of  $\beta S$ .*

*Proof.* (1) Suppose that  $x$  is uniformly recurrent. Then  $T_u(x) = x$  for some  $u \in K(\beta S)$ . Thus for every  $v \in \beta S$ ,  $T_v(x) = T_v T_u(x) = T_{vu}(x)$ ; since  $vu \in K(\beta S)$ , by Lemma 2.1(f),  $T_v(x)$  is uniformly recurrent.

(2) Let  $x \in X$ , let  $p \in U(x)$ , and let  $r \in \beta S$ . By Lemma 2.1(e), pick  $q \in K(\beta S)$  such that  $T_q(T_p(x)) = T_p(x)$ . Then  $T_{rp}(x) = T_r(T_q(T_p(x))) = T_{rqp}(x)$ . Now  $rqp \in K(\beta S)$ , so by Lemma 2.1(f),  $T_{rp}(x)$  is uniformly recurrent.

(3) This is immediate from Lemma 2.1(f).

(4) By (3),  $\bigcap_{x \in X} U(x)$  is nonempty, so by (2)  $\bigcap_{x \in X} U(x)$  is a left ideal of  $\beta S$ , so it is enough to show that  $\bigcap_{x \in X} U(x)$  is a right ideal of  $\beta S$ . So suppose that  $x \in X$ ,  $p \in \bigcap_{x \in X} U(x)$  and  $q \in \beta S$ . Since  $p \in U(T_q(x))$ ,  $T_{pq}(x)$  is uniformly recurrent and so  $pq \in U(x)$ .  $\square$

The statements of Lemma 2.3 below are modifications of basic well known facts that are proved in [2]. (Furstenberg assumes that the phase space is metric, but the proofs given do not use this assumption.) We shall say that a subspace  $Z$  of  $X$  is *invariant* if  $T_s[Z] \subseteq Z$  for every  $s \in S$ . Of course, if  $Z$  is closed and invariant, then  $T_p[Z] \subseteq Z$  for every  $p \in \beta S$ . (Let  $x \in Z$ . Then  $T_s(x) \in Z$  for each  $s \in S$  so  $p\text{-}\lim_{s \in S} T_s(x) \in Z$ .)

**Lemma 2.3.** *Let  $(X, \langle T_s \rangle_{s \in S})$  be a dynamical system. Let  $L$  be a minimal left ideal of  $\beta S$ .*

- (1) *A subspace  $Y$  of  $X$  is minimal among all closed and invariant subsets of  $X$  if and only if there is some  $x \in X$  such that  $Y = \{T_p(x) : p \in L\}$ .*
- (2) *Let  $Y$  be a subspace of  $X$  which is minimal among all closed and invariant subsets of  $X$ . Then every element of  $Y$  is uniformly recurrent.*
- (3) *If  $x \in X$  is uniformly recurrent and  $Y = \{T_p(x) : p \in \beta S\}$ , then  $Y$  is minimal among all closed and invariant subsets of  $X$ .*

(4) If  $x \in X$  is uniformly recurrent, then  $T_p(x)$  is uniformly recurrent for every  $p \in \beta S$ .

*Proof.* (1) Suppose that  $Y$  is a subspace of  $X$  which is minimal among all closed and invariant subsets of  $X$ . Pick  $x \in Y$  and let  $Z = \{T_p(x) : p \in L\}$ . We claim that  $Z$  is a closed and invariant subspace of  $Y$  and is therefore equal to  $Y$ . If  $p \in L$  and  $s \in S$ , then  $T_s(T_p(x)) = T_{sp}(x)$  and  $sp \in L$ , so  $Z$  is invariant and obviously  $Z \subseteq Y$ . To see that  $Z$  is closed, it suffices to show that any net in  $Z$  has a cluster point in  $Z$ . To this end, let  $\langle p_\alpha \rangle_{\alpha \in D}$  be a net in  $L$  and pick a cluster point  $p$  in  $L$  of  $\langle p_\alpha \rangle_{\alpha \in D}$ . Then  $T_p(x)$  is a cluster point of  $\langle T_{p_\alpha}(x) \rangle_{\alpha \in D}$ .

Conversely, let  $x \in X$  and let  $Y = \{T_p(x) : p \in L\}$ . Then  $Y$  is invariant and one sees as above that  $Y$  is closed. We shall show that  $Y$  is minimal among all closed and invariant subsets of  $X$ . To see this, suppose that  $Z$  is a subset of  $Y$  which is closed and invariant. We shall show that  $Y \subseteq Z$ , so let  $y \in Y$  be given. Pick  $z \in Z$ . Then  $y = T_p(x)$  and  $z = T_q(x)$  for some  $p$  and  $q$  in  $L$ . Since  $Lq = L$ , there exists  $r \in L$  such that  $rq = p$ . It follows that  $T_r(z) = y$  and hence that  $y \in Z$  as required.

(2) Let  $Y$  be a subspace of  $X$  which is minimal among all closed and invariant subsets of  $X$  and let  $x \in Y$ . Pick  $y \in X$  such that  $Y = \{T_p(y) : p \in L\}$ . Pick  $p \in L$  such that  $x = T_p(y)$ . By Lemma 2.1(f),  $x$  is uniformly recurrent.

(3) Let  $x$  be a uniformly recurrent point of  $X$  and let  $Y = \{T_p(x) : p \in \beta S\}$ . By Lemma 2.1(b), pick  $q \in L$  such that  $T_q(x) = x$ . By (1) it suffices that  $Y = \{T_p(x) : p \in L\}$ . To see this, let  $y \in Y$  and pick  $p \in \beta S$  such that  $y = T_p(x)$ . Then  $y = T_p(T_q(x)) = T_{pq}(x)$  and  $pq \in L$ .

(4) Let  $x$  be a uniformly recurrent point of  $X$  and let  $Y = \{T_p(x) : p \in \beta S\}$ . By (3)  $Y$  is minimal among all closed and invariant subsets of  $X$  so (2) applies.  $\square$

We conclude this section with a few results about the dynamical system  $(\beta S, \langle \lambda_s \rangle_{s \in S})$ . We observe that, if we define  $\lambda_p : \beta S \rightarrow \beta S$  in this system by  $\lambda_p(q) = \lim_{s \rightarrow p} \lambda_s(q)$ , where  $s$  denotes an element of  $S$ , then  $\lambda_p(q) = pq$  for every  $p$  and  $q$  in  $\beta S$ . So this does not conflict with the previous definition of  $\lambda_p$  given in the introduction.

**Theorem 2.4.** *Let  $S$  be a semigroup and let  $x \in \beta S$ . Statements (a) and (b) are equivalent and imply (c). If  $\beta S$  has a left cancelable element, all three are equivalent.*

(a)  $x \in K(\beta S)$ .

(b)  $x$  is uniformly recurrent in the dynamical system  $(\beta S, \langle \lambda_s \rangle_{s \in S})$ .

(c)  $\beta Sx$  is a minimal left ideal of  $\beta S$ .

*Proof.* To see that (a) implies (b), let  $x \in K(\beta S)$  and let  $u$  be the identity of the group in  $K(\beta S)$  to which  $x$  belongs. Then  $x = \lambda_u(x)$  so by Lemma uniformrecurrence(e),  $x$  is uniformly recurrent.

To see that (b) implies (a), assume that  $x$  is uniformly recurrent. By Lemma 2.1(f) pick  $y \in \beta S$  and  $q \in K(\beta S)$  such that  $\lambda_q(y) = x$ . Then  $x = qy \in K(\beta S)$ .

To see that (a) implies (c), assume that  $x \in K(\beta S)$  and pick the minimal left ideal  $L$  of  $\beta S$  such that  $x \in L$ . Then  $\beta Sx$  is a left ideal of  $\beta S$  contained in  $L$  and so  $\beta Sx = L$ .

Now assume that  $\beta S$  has a left cancelable element  $z$  and that  $\beta Sx$  is a minimal left ideal of  $\beta S$ . Pick an idempotent  $u \in \beta Sx$ . Then  $zx \in \beta Sx$  so by [4, Lemma 1.30],  $zx = zxu$  and therefore  $x = xu \in \beta Sx \subseteq K(\beta S)$ .  $\square$

**Corollary 2.5.** *Let  $S$  be an infinite semigroup and let  $x \in K(\beta S)$ . Then  $U(x) = \beta S$  with respect to the dynamical system  $(\beta S, \langle \lambda_s \rangle_{s \in S})$ .*

*Proof.* By Theorem 2.4,  $x$  is uniformly recurrent, so by Lemma 2.3(4),  $U(x) = \beta S$ .  $\square$

**Corollary 2.6.** *Let  $S$  be a semigroup and let  $p, q \in \beta S$ . Statements (a) and (b) are equivalent and imply statement (c). If  $\beta S$  has a left cancelable element, then all three statements are equivalent.*

- (a)  $qp \in K(\beta S)$ .
- (b)  $q \in U(p)$  with respect to the dynamical system  $(\beta S, \langle \lambda_s \rangle_{s \in S})$ .
- (c)  $\beta Sqp$  is a minimal left ideal of  $\beta S$ .

*Proof.* We have that  $q \in U(p)$  if and only if  $\lambda_q(p)$  is uniformly recurrent and  $\lambda_q(p) = qp$  so Theorem 2.4 applies.  $\square$

It is an old and difficult problem to characterize when  $K(\beta S)$  is prime or when  $\text{cl}K(\beta S)$  is prime. There are trivial situations where the answer is known. For example if  $S$  is left zero or right zero, then so is  $\beta S$  and thus  $K(\beta S) = \beta S$ , and is necessarily prime. It is not known whether  $K(\beta \mathbb{N}, +)$  is prime or  $\text{cl}K(\beta \mathbb{N}, +)$  is prime. (Some partial results were obtained in [3].)

**Corollary 2.7.** *Let  $S$  be a semigroup. The following statements are equivalent.*

- (a) *There exists  $p \in \beta S \setminus K(\beta S)$  such that, with respect to the dynamical system  $(\beta S, \langle \lambda_s \rangle_{s \in S})$ ,  $K(\beta S) \subsetneq U(p)$ .*
- (b)  *$K(\beta S)$  is not prime.*

*Proof.* This is an immediate consequence of Corollary 2.6.  $\square$

### 3 Dynamical systems with phase space $R_{\{0,1\}}$

Throughout this section we assume that  $R$  is a semigroup,  $S$  a subsemigroup of  $R$ , and  $(Z, \langle T_s \rangle_{s \in S})$  is the dynamical system of Lemma 1.2. While our results are valid in this generality, in practice we are interested in just two situations, one in which  $R = S$  and the other in which  $R = S \cup \{e\}$  where  $e$  is a two sided identity adjoined to  $S$ .

Our first results in this section are aimed at showing that for any semigroup  $S$ , there is a dynamical system such that both  $K(\beta S)$  and  $\text{cl}K(\beta S)$  are intersections of sets of the form  $U(x)$ .

**Definition 3.1.** Given  $x \in Z$  we denote the continuous extension of  $x$  from  $\beta S$  to  $\{0,1\}$  by  $\tilde{x}$ .

Of course, for each  $x \in Z$ , each  $p \in \beta S$  and each  $t \in R$ ,  $T_p(x)(t) = p\text{-}\lim_{s \in S} T_s(x)(t) = p\text{-}\lim_{s \in S} x(ts)$  and so  $T_p(x)(t) = \tilde{x}(tp)$ .

**Lemma 3.2.** Let  $x \in Z$ , let  $p \in \beta S$ , and let  $L$  be a minimal left ideal of  $\beta S$ . The following statements are equivalent:

- (a)  $p \in U(x)$ .
- (b) There exists  $q \in L$  such that  $\tilde{x}(tp) = \tilde{x}(tqp)$  for all  $t \in R$ .
- (c) There exists an idempotent  $q \in L$  such that  $\tilde{x}(tp) = \tilde{x}(tqp)$  for all  $t \in R$ .

*Proof.* To see that (a) implies (c), assume that  $T_p(x)$  is uniformly recurrent. By Lemma 2.1(c), pick an idempotent  $q \in L$  such that  $T_q(T_p(x)) = T(p)(x)$ . Then  $T_{qp}(x) = T_p(x)$  so as noted above, for all  $t \in R$ ,  $\tilde{x}(tqp) = \tilde{x}(tp)$ .

Trivially (c) implies (b). To see that (b) implies (a), pick  $q \in L$  such that  $\tilde{x}(tp) = \tilde{x}(tqp)$  for all  $t \in R$ . Then  $T_p(x) = T_{qp}(x) = T_q(T_p(x))$ , so by Lemma 2.1(b),  $T_p(x)$  is uniformly recurrent.  $\square$

**Lemma 3.3.** Let  $x \in Z$  and let  $p \in \beta S$ . Then  $p \in U(x)$  if and only if for every minimal left ideal  $L$  of  $\beta S$  and every  $F \in \mathcal{P}_f(R)$ , there exists  $q_F \in L$  such that for all  $t \in F$ ,  $\tilde{x}(tp) = \tilde{x}(tq_Fp)$ .

*Proof.* The necessity is an immediate consequence of Lemma 3.2(b).

For the sufficiency, let  $L$  be a minimal left ideal of  $\beta S$ . For each  $F \in \mathcal{P}_f(R)$ , pick  $q_F \in L$  as guaranteed. Direct  $\mathcal{P}_f(R)$  by agreeing that  $F < G$  if and only if  $F \subseteq G$ . Pick a cluster point  $q \in L$  of the net  $\langle q_F \rangle_{F \in \mathcal{P}_f(R)}$ . It is then routine to show that for all  $t \in R$ ,  $\tilde{x}(tqp) = \tilde{x}(tp)$  so that by Lemma 3.2(b),  $p \in U(x)$ .  $\square$

**Theorem 3.4.** (1)  $K(\beta S) \subseteq \bigcap_{x \in Z} U(x)$ .



- (2) If  $p \in \bigcap_{x \in Z} U(x)$ , then, for every minimal left ideal  $L$  of  $\beta S$ ,  $\beta Sp = Lp$  and so  $\beta Sp$  is a minimal left ideal of  $\beta S$ .
- (3) If  $R$  contains a left cancelable element, then  $K(\beta S) = \bigcap_{x \in Z} U(x)$ . In particular, if  $R$  has a left identity, then  $K(\beta S) = \bigcap_{x \in Z} U(x)$ .

*Proof.* (1)  $K(\beta S) \subseteq \bigcap_{x \in Z} U(x)$  by Corollary 2.2(3).

(2) Assume that  $p \in \bigcap_{x \in Z} U(x)$ . Let  $L$  be a minimal left ideal of  $\beta S$ . We shall show that, for every  $t \in R$ ,  $tp \in tLp$ . To see this, assume the contrary. Then for some  $t \in R$ , there exists  $A \subseteq R$  such that  $A \in tp$  and  $\overline{A} \cap tLp = \emptyset$ . Let  $x = \chi_A$ . So  $\tilde{x}$  is the characteristic function of  $\overline{A}$ . Since  $p \in U(x)$ , it follows from Lemma 3.2 that  $\tilde{x}(tp) = \tilde{x}(tqp)$  for some  $q \in L$ . However,  $\tilde{x}(tp) = 1$  and  $\tilde{x}(tqp) = 0$ . This contradiction establishes that  $tp \in tLp$  for every  $t \in R$ . In particular,  $\beta Sp = cl_{\beta S} Sp \subseteq Lp$ . So  $\beta Sp \subseteq Lp$ . By [4, Theorem 1.46],  $Lp$  is a minimal left ideal of  $\beta S$ , and so  $\beta Sp = Lp$ .

(3) Now suppose that  $R$  contains a left cancelable element  $t$  and let  $p \in \bigcap_{x \in Z} U(x)$ . Since  $t$  is left cancelable in  $\beta R$  by [4, Lemma 8.1] and  $tp = tqp$  for some  $q \in L$ , it follows that  $p = qp \in K(\beta S)$ .  $\square$

Recall that a subset  $A$  of a semigroup  $S$  is *piecewise syndetic* if and only if there is some  $G \in \mathcal{P}_f(S)$  such that for every  $F \in \mathcal{P}_f(S)$ , there is some  $x \in S$  with  $Fx \subseteq \bigcup_{t \in G} t^{-1}A$ . The important fact about piecewise syndetic sets is that they are the subsets of  $S$  whose closure meets  $K(\beta S)$ , [4, Theorem 4.40].

**Definition 3.5.**  $\Omega = \Omega_Z = \{x \in Z : \overline{x^{-1}[\{1\}] \cap S} \cap K(\beta S) = \emptyset\}$ .

Thus  $\Omega = \{x \in Z : x^{-1}[\{1\}] \cap S \text{ is not piecewise syndetic in } S\}$ . Note that, since  $K(\beta S)$  is usually not topologically closed, we have by Theorem 3.4 that not all sets of the form  $U(x)$  are closed.

**Definition 3.6.** Let  $x \in Z$ .  $N(x) = \{p \in \beta S : (\forall t \in R)(T_p(x)(t) = 0)\}$ .

**Lemma 3.7.** Let  $x \in Z$ . Then  $N(x)$  is closed and  $N(x) \subseteq U(x)$ . If  $N(x) = U(x)$ , then  $x \in \Omega$ . If  $S$  is a left ideal of  $R$ , then  $N(x) = U(x)$  if and only if  $x \in \Omega$ .

*Proof.* To see that  $N(x)$  is closed, let  $p \in \beta S \setminus N(x)$ , pick  $t \in R$  such that  $T_p(x)(t) = 1$ , and let  $A = \{s \in S : T_s(x)(t) = 1\}$ . Then  $A \in p$  and  $\overline{A} \cap N(x) = \emptyset$ .

If  $T_p(x)$  is constantly equal to 0 on  $R$ , then  $T_p(x)$  is uniformly recurrent and thus  $p \in U(x)$ .

Let  $A = x^{-1}[\{1\}] \cap S$ .

First assume that  $N(x) = U(x)$  and suppose that  $x \notin \Omega$ . Since  $\overline{A} \cap K(\beta S) \neq \emptyset$ , pick  $p \in \overline{A} \cap K(\beta S)$ . By Corollary 2.2(3),  $p \in U(x)$  and so for all  $t \in R$ ,  $T_p(x)(t) = 0$ . Since  $K(\beta S)$  is a union of groups, there exists  $q \in K(\beta S)$  such

that  $qp = p$ . Pick  $t \in S$  such that  $t^{-1}A \in p$ . Also  $T_p(x)(t) = 0$  so  $\{s \in S : x(ts) = 0\} \in p$ . Pick  $s \in t^{-1}A$  such that  $x(ts) = 0$ , a contradiction.

Now assume that  $S$  is a left ideal in  $R$ . Let  $x \in \Omega$  and let  $p \in U(x)$ . We claim that  $p \in N(x)$ . To see this, suppose we have some  $t \in R$  such that  $T_p(x)(t) = 1$ . By Lemma 3.2, there exists an idempotent  $q \in K(\beta S)$  such that  $\tilde{x}(tqp) = 1$ . By [4, Theorem 2.17],  $\beta S$  is a left ideal of  $\beta R$  so  $tqp \in \beta S$  and so  $A \in tqp = tqqp$ . Thus there is some  $s \in S$  such that  $tsqp \in \overline{A}$ . Since  $ts \in S$ ,  $tsqp \in K(\beta S)$ , a contradiction.  $\square$

**Lemma 3.8.** *Let  $p \in \bigcap_{x \in \Omega} U(x)$  and let  $t \in R$ . If  $tp \in \beta S$ , then  $tp \in \text{cl}K(\beta S)$ . In particular,  $\beta Sp \subseteq \text{cl}K(\beta S)$ .*

*Proof.* Assume that  $tp \in \beta S \setminus \text{cl}(K\beta S)$ . We can choose  $A \in tp$  such that  $A \subseteq S$  and  $\overline{A} \cap K(\beta S) = \emptyset$ . Let  $x$  be the characteristic function of  $A$  in  $R$ , so that  $x \in \Omega$  and hence  $p \in U(x)$ . Observe that  $\tilde{x}$  is the characteristic function of  $\text{cl}_{\beta R}(A)$  in  $\beta R$  and that  $\text{cl}_{\beta R}(A) \subseteq \beta S$ , because  $\beta S$  is clopen in  $\beta R$ . Since  $\tilde{x}(tp) = 1$ , it follows from Lemma 3.2(b) that there exists  $q \in K(\beta S)$  such that  $\tilde{x}(tqp) = 1$ , and so  $A \in tqp$ . Now  $\{r \in \beta S : tqr \in \beta S\}$  is non-empty and is a right ideal of  $\beta S$ . There exists an idempotent  $u$  in the intersection of this right ideal with the left ideal  $\beta Sq$  of  $\beta S$ . Since  $q \in \beta Su$ ,  $qu = q$ . So  $tqp = tqup \in K(\beta S)$ , because  $tqu \in \beta S$  and  $u \in \beta Sq \subseteq K(\beta S)$ . This contradicts the assumption that  $\overline{A} \cap K(\beta S) = \emptyset$ .  $\square$

**Corollary 3.9.** *Each of the following statements implies that  $\bigcap_{x \in \Omega} U(x) \subseteq \text{cl}K(\beta S)$ .*

- (a) *There exists  $e \in R$  such that  $es = s$  for every  $s \in S$ .*
- (b)  *$S$  contains a left cancelable element.*

*Proof.* It follows from Lemma 3.8 that (a) implies that  $\bigcap_{x \in \Omega} U(x) \subseteq \text{cl}K(\beta S)$ . So assume that  $s$  is a left cancelable element in  $S$  and let  $p \in \bigcap_{x \in \Omega} U(x)$ . By [4, Lemma 8.1],  $s$  is left cancelable in  $\beta S$ . By Lemma 3.8,  $sp \in \text{cl}K(\beta S)$ . Now  $s\beta S = \overline{sS}$  is clopen in  $\beta S$ . So  $sp \in \text{cl}(K(\beta S) \cap s\beta S)$ . We claim that, if  $q \in K(\beta S) \cap s\beta S$ , then  $q \in sK(\beta S)$ . To see this, suppose that  $q \in K(\beta S)$  and that  $q = sv$  for some  $v \in \beta S$ . There is an idempotent  $u \in K(\beta S)$  for which  $qu = q$ . This implies that  $sv = svu$  and hence that  $v = vu \in K(\beta S)$ . So  $sp \in \text{cl}(sK(\beta S)) = s\text{cl}K(\beta S)$  and hence  $p \in \text{cl}K(\beta S)$ .  $\square$

**Corollary 3.10.** *Assume that  $S$  is a left ideal of  $R$ . Then each of the hypotheses (a) and (b) of Corollary 3.9 implies that  $\bigcap_{x \in \Omega} U(x) = \text{cl}K(\beta S)$ .*

*Proof.* Assume that one of the hypotheses of Corollary 3.9 holds. Then

$$\bigcap_{x \in \Omega} U(x) \subseteq \text{cl}K(\beta S).$$

To see that  $\text{cl}K(\beta S) \subseteq \bigcap_{x \in \Omega} U(x)$ , let  $x \in \Omega$  be given. By Lemma 3.7,  $U(x) = N(x)$  and so  $U(x)$  is closed. By Corollary 2.2(3),  $K(\beta S) \subseteq U(x)$  and hence  $\text{cl}K(\beta S) \subseteq U(x)$ .  $\square$

For the statement of the following corollary we depart from our standing assumptions about  $R$ ,  $S$ , and  $(Z, \langle T_s \rangle_{s \in S})$ .

**Corollary 3.11.** *Let  $S$  be a semigroup. There exist a dynamical system  $(X, \langle T_s \rangle_{s \in S})$  and a subset  $M$  of  $X$  such that  $K(\beta S) = \bigcap_{x \in X} U(x)$  and  $\text{cl}K(\beta S) = \bigcap_{x \in M} U(x)$ .*

*Proof.* If  $S$  has a left identity, let  $R = S$ . Otherwise, let  $R = S \cup \{e\}$  where  $e$  is an identity adjoined to  $S$ . The conclusion then follows from Theorem 3.4 and Corollary 3.10.  $\square$

In the proof of the above corollary, we could have simply let  $R = S \cup \{e\}$  where  $e$  is an identity adjoined to  $S$ , regardless of whether  $S$  has a left identity, as was done in [4, Theorem 19.27] to produce a dynamical system for any semigroup  $S$  establishing the equivalence of the notions of *central* and *dynamically central*. We shall investigate the relationship between the systems with phase space  $X = {}^R\{0, 1\}$  and  $Y = {}^S\{0, 1\}$  in the next section.

We note that it is possible that  $\bigcap_{x \in Z} U(x) \neq K(\beta S)$  and there is no subset  $M$  of  $Z$  such that  $\bigcap_{x \in M} U(x) = \text{cl}K(\beta S)$ . To see this, let  $S$  be an infinite zero semigroup. That is, there is an element  $0 \in S$  such that  $st = 0$  for all  $s$  and  $t$  in  $S$ . Then  $pq = 0$  for all  $p$  and  $q$  in  $\beta S$  and so  $\text{cl}K(\beta S) = K(\beta S) = \{0\}$ . Let  $R = S$ . Given  $x \in T$ , if  $a = x(0)$ , then for all  $p \in \beta S$ ,  $T_p(x)$  is constantly equal to  $a$  and so  $T_p(x)$  is uniformly recurrent. That is, for any  $x \in Z$ ,  $U(x) = \beta S$ .

In [1] it was shown that  $\text{cl}K(\beta \mathbb{N})$  is the intersection of all of the closed two sided ideals that strictly contain it. In a similar vein, we would like to show that each  $U(x)$  properly contains  $K(\beta S)$ . One cannot hope for this to hold in general. For example, as we have already noted, if  $S$  is either left zero or right zero then so is  $\beta S$  and then  $K(\beta S) = \beta S$ . Results establishing that  $U(x)$  properly contains  $K(\beta S)$  require some weak cancellation assumptions.

**Definition 3.12.** Let  $S$  be a semigroup and let  $A \subseteq S$ .

- (a)  $A$  is a *left solution set* if and only if there exist  $u$  and  $v$  in  $S$  such that  $A = \{x \in S : ux = v\}$ .
- (b)  $A$  is a *right solution set* if and only if there exist  $u$  and  $v$  in  $S$  such that  $A = \{x \in S : xu = v\}$ .

As is standard, we denote by  $\omega$  the first infinite ordinal, which is also the first infinite cardinal. That is,  $\omega = \aleph_0$ .

**Definition 3.13.** Let  $S$  be a semigroup with  $|S| = \kappa \geq \omega$ .

- (a)  $S$  is *weakly left cancellative* if and only if every left solution set in  $S$  is finite.
- (b)  $S$  is *weakly right cancellative* if and only if every right solution set in  $S$  is finite.
- (c)  $S$  is *weakly cancellative* if and only if  $S$  is both weakly left cancellative and weakly right cancellative.
- (d)  $S$  is *very weakly left cancellative* if and only if the union of any set of fewer than  $\kappa$  left solution sets has cardinality less than  $\kappa$ .
- (e)  $S$  is *very weakly right cancellative* if and only if the union of any set of fewer than  $\kappa$  right solution sets has cardinality less than  $\kappa$ .
- (f)  $S$  is *very weakly cancellative* if and only if  $S$  is both very weakly left cancellative and very weakly right cancellative.

Given a set  $X$  and a cardinal  $\kappa$ , we let  $U_\kappa(X)$  be the set of  $\kappa$ -uniform ultrafilters on  $X$ . That is,  $U_\kappa(X) = \{p \in \beta X : (\forall A \in p)(|A| \geq \kappa)\}$ .

**Theorem 3.14.** *Assume that  $|R| = |S| = \kappa \geq \omega$ ,  $S$  is very weakly cancellative, and has the property that  $|\{e \in S : (\exists s \in S)(es = s)\}| < \kappa$ . Then for all  $x \in Z$ ,  $U(x) \cap U_\kappa(S) \setminus \text{cl}K(\beta S) \neq \emptyset$ .*

*Proof.* Let  $E = \{e \in S : (\exists s \in S)(es = s)\}$ . Let  $x \in Z$  and pick  $q \in K(\beta S)$ . Let  $y = T_q(x)$ . By Corollary 2.2(3),  $y$  is uniformly recurrent. For each  $F \in \mathcal{P}_f(R)$ , let  $B_F = \{s \in S : (\forall t \in F)(x(ts) = y(t))\}$ . Since

$$B_F = \{s \in S : T_s(x) \in \bigcap_{t \in F} \pi_t^{-1}[\{y(t)\}]\},$$

we have  $B_F \in q$ . By [4, Lemma 6.34.3],  $K(\beta S) \subseteq U_\kappa(S)$  and so  $|B_F| = \kappa$ . Note that if  $F \subseteq H$ , then  $B_H \subseteq B_F$ .

Enumerate  $\mathcal{P}_f(R)$  as  $\langle F_\alpha \rangle_{\alpha < \kappa}$ . Choose  $t_0 \in B_{F_0} \setminus E$ . Let  $0 < \alpha < \kappa$  and assume that we have chosen  $\langle t_\delta \rangle_{\delta < \alpha}$  satisfying the following inductive hypotheses.

- (1) For each  $\delta < \alpha$ ,  $t_\delta \in B_{F_\delta}$ .
- (2) For each  $\delta < \alpha$ ,  $FP(\langle t_\beta \rangle_{\beta \leq \delta}) \cap E = \emptyset$ .
- (3) For each  $\delta < \alpha$ , if  $\delta > 0$ , then  $t_\delta \notin FP(\langle t_\beta \rangle_{\beta < \delta})$ .
- (4) For each  $\delta < \alpha$ , if  $\delta > 0$ ,  $s \in FP(\langle t_\beta \rangle_{\beta < \delta})$ , and  $\gamma < \delta$ , then  $st_\delta \neq t_\gamma$ .

The hypotheses are satisfied for  $\delta = 0$ . Let

$$\begin{aligned} M_0 &= \{t \in S : (\exists H \in \mathcal{P}_f(R))((\prod_{\beta \in H} t_\beta)t \in E)\} \text{ and let} \\ M_1 &= \{t \in S : (\exists s \in FP(\langle t_\beta \rangle_{\beta < \alpha}))(\exists \gamma < \alpha)(st = t_\gamma)\}. \end{aligned}$$

Note that  $|FP(\langle t_\beta \rangle_{\beta < \alpha})| \leq |\mathcal{P}_f(\alpha)| < \kappa$ . Also, given  $H \in \mathcal{P}_f(\alpha)$  and  $s \in E$ ,  $\{t \in S : (\prod_{\beta \in H} t_\beta)t = s\}$  is a left solution set so  $|M_0| < \kappa$ . Note also that, given  $s \in FP(\langle t_\beta \rangle_{\beta < \alpha})$  and  $\gamma < \alpha$ ,  $\{t \in S : st = t_\gamma\}$  is a left solution set so  $|M_1| < \kappa$ . Thus we may choose  $t_\alpha \in B_{F_\alpha} \setminus (E \cup FP(\langle t_\beta \rangle_{\beta < \alpha}) \cup M_0 \cup M_1)$ . The induction hypotheses are satisfied for  $\alpha$ .

Let  $B = \{t_\alpha : \alpha < \kappa\}$  and let  $C = \bigcap_{\alpha < \kappa} clFP(\langle t_\beta \rangle_{\alpha < \beta < \kappa})$ . By [4, Theorem 4.20],  $C$  is a compact subsemigroup of  $\beta S$ . We claim that  $\overline{B} \cap K(C) = \emptyset$ . Suppose instead that we have  $p \in \overline{B} \cap K(C)$ . Pick  $r \in K(C)$  such that  $p = pr$ . (By [4, Lemma 1.30], an idempotent in the minimal left ideal  $L$  of  $C$  in which  $p$  lies will do.) Let  $D = \{s \in S : s^{-1}B \in r\}$ . Then  $D \in p$  so  $D \cap B \neq \emptyset$  so pick  $\alpha < \kappa$  such that  $t_\alpha^{-1}B \in r$ . Then  $\overline{t_\alpha^{-1}B} \cap FP(\langle t_\beta \rangle_{\alpha < \beta < \kappa}) \neq \emptyset$  so pick finite  $H \subseteq \{\beta : \alpha < \beta < \kappa\}$  such that  $\prod_{\beta \in H} t_\beta \in t_\alpha^{-1}B$ . Pick  $\gamma < \kappa$  such that  $t_\alpha \prod_{\beta \in H} t_\beta = t_\gamma$ . Let  $\max H = \mu$  and let  $K = H \setminus \{\mu\}$ . If  $K = \emptyset$ , then  $t_\alpha t_\mu = t_\gamma$ . If  $K \neq \emptyset$ , then  $t_\alpha (\prod_{\beta \in K} t_\beta) t_\mu = t_\gamma$ . If  $\gamma > \mu$  we get a contradiction to hypothesis (3). If  $\mu = \gamma$  we either get  $t_\alpha \in E$  or  $t_\alpha \prod_{\beta \in K} t_\beta \in E$ , contradicting hypothesis (2). If  $\gamma < \mu$  we get a contradiction to hypothesis (4). Thus  $\overline{B} \cap K(C) = \emptyset$  as claimed.

Now we claim that  $\overline{B} \cap K(\beta S) = \emptyset$ . Suppose instead we have  $p \in \overline{B} \cap K(\beta S)$ . By [4, Lemma 6.34.3] we have that  $p \in U_\kappa(S)$  and consequently,  $p \in C$ . Thus  $K(\beta S) \cap C \neq \emptyset$  and so, by [4, Theorem 1.65],  $K(C) = K(\beta S) \cap C$ , contradicting the fact that  $\overline{B} \cap K(C) = \emptyset$ . Since  $\overline{B}$  is clopen, we thus have  $\overline{B} \cap clK(\beta S) = \emptyset$ .

Now let  $\mathcal{C} = \{B_F : F \in \mathcal{P}_f(S)\} \cup \{B\}$ . We claim that  $\mathcal{C}$  has the  $\kappa$ -uniform finite intersection property. To see this, let  $\mathcal{F} \in \mathcal{P}_f(\mathcal{P}_f(S))$  and let  $H = \bigcup \mathcal{F}$ . If  $\delta < \kappa$  and  $H \subseteq F_\delta$ , then  $t_\delta \in B \cap \bigcap_{F \in \mathcal{F}} B_F$ . Since  $|\{\delta < \kappa : H \subseteq F_\delta\}| = |\{F \in \mathcal{P}_f(S) : H \subseteq F\}| = \kappa$ , we have that  $|\bigcap \mathcal{C}| = \kappa$  as required. Pick by [4, Corollary 3.14]  $p \in U_\kappa(S)$  such that  $\mathcal{C} \subseteq p$ .

Since  $B_F \in p$  for each  $F \in \mathcal{P}_f(R)$ , we have  $T_p(x) = y$  so  $p \in U(x)$ . Since  $B \in p$ ,  $p \notin clK(\beta S)$ .  $\square$

**Corollary 3.15.** *Assume that  $|R| = |S| = \kappa \geq \omega$  and that  $S$  is right cancellative and very weakly left cancellative. Then for all  $x \in Z$ ,  $U(x) \cap U_\kappa(S) \setminus clK(\beta S) \neq \emptyset$ .*

*Proof.* Let  $E = \{e \in S : (\exists s \in S)(es = s)\}$ . It suffices to show that  $|E| < \kappa$ . Pick  $x \in S$ . Given  $e \in E$  and  $s \in S$  such that  $es = s$ , we have that  $xes = xs$  so  $xe = x$ . Thus  $E$  is contained in the left solution set  $\{t \in S : xt = x\}$ .  $\square$

**Corollary 3.16.** *Assume that  $|R| = |S| = \kappa \geq \omega$ , that  $S$  is very weakly cancellative, that  $S$  has a member  $e$  such that  $es = s$  for all  $s \in S$ , and  $|\{e \in S : (\exists s \in S)(es = s)\}| < \kappa$ . Then  $K(\beta S) = \bigcap_{x \in Z} U(x)$  and for each  $x \in Z$ ,  $U(x)$  properly contains  $K(\beta S)$ .*

*Proof.* This is an immediate consequence of Theorems 3.4 and 3.14.  $\square$

**Corollary 3.17.** *Assume that  $S$  is a left ideal of  $R$ ,  $|R| = |S| = \kappa \geq \omega$ ,  $S$  is very weakly cancellative,  $S$  has a member  $e$  such that  $es = s$  for all  $s \in S$ , and  $|\{e \in S : (\exists s \in S)(es = s)\}| < \kappa$ . Then  $\text{cl}K(\beta S) = \bigcap_{x \in \Omega} U(x)$  and for each  $x \in \Omega$ ,  $U(x)$  properly contains  $\text{cl}K(\beta S)$ .*

*Proof.* By Corollary 3.10  $\text{cl}K(\beta S) = \bigcap_{x \in \Omega} U(x)$ . By Theorem 3.14, for each  $x \in \Omega$ ,  $U(x)$  properly contains  $\text{cl}K(\beta S)$ .  $\square$

## 4 Relations between systems with phase spaces $X$ and $Y$

Throughout this section we will let  $S$  be an arbitrary semigroup and let  $Q = S \cup \{e\}$ , where  $e$  is an identity adjoined to  $S$ , even if  $S$  already has an identity. We will let  $(X, \langle T_{X,s} \rangle_{s \in S})$  be the dynamical system of Lemma 1.2 determined by  $R = Q$  let  $(Y, \langle T_{Y,s} \rangle_{s \in S})$  be the dynamical system of Lemma 1.2 determined by  $R = S$ . For  $x \in X$  we will let  $U_X(x) = \{p \in \beta S : T_{X,p}(x) \text{ is uniformly recurrent}\}$  and let  $U_Y(x) = \{p \in \beta S : T_{Y,p}(x) \text{ is uniformly recurrent}\}$ .

We have from the results of the previous section that for any semigroup  $S$ ,  $K(\beta S) = \bigcap_{x \in X} U_X(x)$  and  $\text{cl}K(\beta S) = \bigcap_{x \in \Omega_X} U_X(x)$ . We are interested in determining when the corresponding assertions hold with respect to  $Y$ . Of course, the simplest situation in which they do is when for each  $x \in X$ ,  $U_X(x) = U_Y(x|_S)$  so we address this problem first, beginning with the following simple observation.

**Lemma 4.1.** *Let  $x \in X$ . Then  $U_X(x) \subseteq U_Y(x|_S)$ .*

*Proof.* Let  $y = x|_S$  and note that  $\tilde{y}$  is the restriction of  $\tilde{x}$  to  $\beta S$ . Let  $L$  be a minimal left ideal of  $\beta S$ . By Lemma 3.2,  $p \in U_X(x)$  if and only if there exists  $q \in L$ , such that  $\tilde{x}(tp) = \tilde{x}(tqp)$  for all  $t \in Q$ . And  $p \in U_Y(x|_S)$  if and only if there exists  $q \in L$  such that  $\tilde{y}(tp) = \tilde{y}(tqp)$  for all  $t \in S$ .  $\square$

**Theorem 4.2.** *The following statements are equivalent.*

- (a) *For all  $x \in X$ ,  $U_X(x) = U_Y(x|_S)$ .*
- (b) *There do not exist  $p \in \beta S$  and  $x \in X$  such that  $T_{X,p}(x)$  is the characteristic function of  $\{e\}$  in  $X$ .*
- (c) *For every  $p \in \beta S$ ,  $p \in \beta Sp$ .*

*Proof.* Assume that (a) holds and suppose we have  $p \in \beta S$  and  $x \in X$  such that  $T_{X,p}(x)$  is the characteristic function of  $\{e\}$  in  $X$ . Then  $T_{Y,p}(x|_S)$  is constantly 0 so  $p \in U_Y(x|_S)$ . But  $V = \{u \in X : w(e) = 1\}$  is a neighborhood of  $w = T_{X,p}(x)$  in  $X$ , while  $\{s \in S : T_{X,s}(w) \in V\} = \emptyset$ , so  $p \notin U_X(x)$ .

To see that (b) implies (c), assume that (b) holds and suppose that we have some  $p \in \beta S$  such that  $p \notin \beta Sp$ . Since  $\beta Sp = \rho_p[\beta S]$ ,  $\beta Sp$  is closed. Pick  $A \in p$  such that  $\overline{A} \cap \beta Sp = \emptyset$ . Let  $x$  be the characteristic function of  $A$  in  $X$ . First let  $s \in S$ . Then  $sp \notin \overline{A}$  so  $s^{-1}(S \setminus A) \in p$  so to see that  $T_{X,p}(s) = 0$ , it suffices to observe that  $s^{-1}(S \setminus A) \subseteq \{t \in S : T_{X,t}(x)(s) = 0\}$ . Since  $A \in p$  and for  $t \in A$ ,  $T_{X,t}(x)(e) = x(t) = 1$ , we have that  $T_{X,p}(x)(e) = 1$ .

By Lemma 4.1, we have  $U_X(x) \subseteq U_Y(x|_S)$  for all  $x \in X$ , so to show that (c) implies (a), it suffices to let  $x \in X$ , let  $p \in U_Y(x|_S)$ , assume that  $p \in \beta Sp$ , and show that  $p \in U_X(x)$ . By Lemma 3.3, it suffices to let  $L$  be a minimal left ideal of  $\beta S$  and let  $F \in \mathcal{P}_f(Q)$  and show that there is some  $q \in L$  such that  $\tilde{x}(tp) = \tilde{x}(tqp)$  for every  $t \in F$ . For  $t \in F$ , let  $B_t = \{s \in S : x(ts) = \tilde{x}(tp)\}$ . Then  $\bigcap_{t \in F} B_t \in p$  and  $p \in \beta Sp = \text{cl}(Sp)$  so pick  $v \in S$  such that  $\bigcap_{t \in F} B_t \in vp$ . Let  $y = x|_S$ . Now  $Fv \in \mathcal{P}_f(S)$  so pick by Lemma 3.3  $q \in L$  such that for all  $t \in F$ ,  $\tilde{y}(tvp) = \tilde{y}(tvqp)$ . Let  $q' = vq$  and note that  $q' \in L$ . Let  $t \in F$  be given. Then  $B_t \in vp$  so  $\tilde{x}(tvp) = \tilde{x}(tp)$  and thus  $\tilde{x}(tp) = \tilde{y}(tvp) = \tilde{y}(tvqp) = \tilde{x}(tq'p)$ .  $\square$

**Corollary 4.3.** *If for all  $p \in \beta S$ ,  $p \in \beta Sp$ , then  $K(\beta S) = \bigcap_{x \in Y} U_Y(x)$  and  $\text{cl}K(\beta S) = \bigcap_{x \in \Omega_Y} U_Y(x)$ .*

*Proof.* The first assertion is an immediate consequence of Theorems 3.4 and 4.2. The second assertion follows from Corollary 3.10 and Theorem 4.2.  $\square$

We have already mentioned the problem of determining whether  $K(\beta S)$  or  $\text{cl}K(\beta S)$  is prime. Recall that an ideal  $I$  in a semigroup is *semiprime* if and only if whenever  $ss \in I$ , one must have  $s \in I$ .

**Corollary 4.4.** (1) *If  $K(\beta S) \neq \bigcap_{x \in Y} U_Y(x)$ , then  $K(\beta S)$  is not semiprime.*

(2) *If  $\text{cl}K(\beta S) \neq \bigcap_{x \in \Omega_Y} U_Y(x)$ , then  $\text{cl}K(\beta S)$  is not semiprime.*

*Proof.* (1) If  $p \in \bigcap_{x \in Y} U_Y(x) \setminus K(\beta S)$ , then  $pp \in \beta Sp$  and by Theorem 3.4,  $\beta Sp \subseteq K(\beta S)$ .

(2) If  $p \in \bigcap_{x \in \Omega_Y} U_Y(x) \setminus \text{cl}K(\beta S)$ , then  $pp \in \beta Sp$  and by Lemma 3.8,  $\beta Sp \subseteq \text{cl}K(\beta S)$ .  $\square$

By virtue of Theorem 4.2 we are interested in knowing when there is some  $p \in \beta S$  such that  $p \notin \beta Sp$ .

**Lemma 4.5.** *Let  $p \in \beta S$ . Then  $p \notin \beta Sp$  if and only if there exists  $A \subseteq S$  such that for all  $x \in S$ ,  $x^{-1}A \in p$  and  $A \notin p$ .*

*Proof.* Let  $C(p) = \{A \subseteq S : (\forall x \in S)(x^{-1}A \in p)\}$ . By [4, Theorem 6.18],  $p \in \beta Sp$  if and only if  $C(p) \subseteq p$ .  $\square$

**Theorem 4.6.** *Assume that  $|S| = \kappa \geq \omega$ . There exists  $p \in \beta S$  such that  $p \notin \beta Sp$  if and only if there exists  $\langle y_F \rangle_{F \in \mathcal{P}_f(S)}$  in  $S$  such that  $\{y_F : F \in \mathcal{P}_f(S)\} \cap \bigcup \{Fy_F : F \in \mathcal{P}_f(S)\} = \emptyset$ .*

*Proof.* Necessity. Pick  $p \in \beta S$  such that  $p \notin \beta Sp$ . By Lemma 4.5, pick  $A \subseteq S$  such that for all  $x \in S$ ,  $x^{-1}A \in p$  and  $A \notin p$ . For  $F \in \mathcal{P}_f(S)$  pick  $y_F \in (S \setminus A) \cap \bigcap_{x \in F} x^{-1}A$ .

Sufficiency. Let  $A = \bigcup \{Fy_F : F \in \mathcal{P}_f(S)\}$ . Then  $\{S \setminus A\} \cup \{x^{-1}A : x \in S\}$  has the finite intersection property so pick  $p \in \beta S$  such that  $\{S \setminus A\} \cup \{x^{-1}A : x \in S\} \subseteq p$ . By Lemma 4.5,  $p \notin \beta Sp$ .  $\square$

One of the assumptions in the following corollary is that  $S^* = \beta S \setminus S$  is a right ideal of  $\beta S$ . A (not very simple) characterization of when  $S^*$  is a right ideal of  $\beta S$  is given in [4, Theorem 4.32]. By [4, Corollary 4.33 and Theorem 4.36] it is sufficient that  $S$  be either right cancellative or weakly cancellative.

**Corollary 4.7.** *Assume that  $|S| = \kappa \geq \omega$  and assume that*

$$|S \setminus \{t \in S : (\exists s \in S)(st = t)\}| = \kappa.$$

*If either  $S^*$  is a right ideal of  $\beta S$  or  $S$  is very weakly left cancellative, then there exists  $p$  in  $\beta S$  such that  $p \notin \beta Sp$ .*

*Proof.* Assume first that  $S^*$  is a right ideal of  $\beta S$ , and pick  $t \in S$  such that there is no  $s \in S$  with  $st = t$ . Then  $t \notin St$  and  $t \notin S^*t$ .

Now assume that  $S$  is very weakly left cancellative. Enumerate  $\mathcal{P}_f(S)$  as  $\langle F_\alpha \rangle_{\alpha < \kappa}$ . By Theorem 4.6, it suffices to produce  $\langle t_\alpha \rangle_{\alpha < \kappa}$  in  $S$  such that  $\{t_\alpha : \alpha < \kappa\} \cap \bigcup \{F_\alpha t_\alpha : \alpha < \kappa\} = \emptyset$ .

Let  $E = \{t \in S : (\exists s \in S)(st = t)\}$ . Pick  $t_0 \in S \setminus E$ . Let  $0 < \alpha < \kappa$  and assume we have chosen  $\langle t_\delta \rangle_{\delta < \alpha}$  in  $S \setminus E$  such that if  $\delta > 0$ , then  $t_\delta \notin \bigcup_{\mu < \delta} F_\mu t_\mu$  and for each  $x \in F_\delta$ ,  $xt_\delta \notin \{t_\mu : \mu < \delta\}$ .

For  $x \in S$  and  $\mu < \alpha$ , let  $H_{x,\mu} = \{t \in S : xt = t_\mu\}$ . Then each  $H_{x,\mu}$  is a left solution set so  $|\bigcup \{H_{x,\mu} : x \in F_\alpha \text{ and } \mu < \alpha\}| < \kappa$ . Pick

$$t_\alpha \in S \setminus (E \cup \bigcup \{H_{x,\mu} : x \in F_\alpha \text{ and } \mu < \alpha\} \cup \bigcup_{\mu < \alpha} F_\mu t_\mu).$$

Suppose we have some  $\mu < \kappa$  such that  $t_\mu \in \bigcup \{F_\alpha t_\alpha : \alpha < \kappa\}$  and pick  $\alpha < \kappa$  and  $x \in F_\alpha$  such that  $t_\mu = xt_\alpha$ . Then  $\alpha \neq \mu$  because  $t_\alpha \notin E$ . If  $\alpha < \mu$ , we would have  $t_\mu \in F_\alpha t_\alpha$ . So we must have  $\mu < \alpha$ . But then  $t_\alpha \in H_{x,\mu}$ , a contradiction.  $\square$

We conclude this section by exhibiting a sufficient condition which guarantees  $K(\beta S) = \bigcap_{x \in Y} U_Y(x)$ . We shall see that this does not require equality between  $U_X(x)$  and  $U_Y(x|_S)$  for all  $x \in X$ .

**Theorem 4.8.** *Assume that for all  $p \in \bigcap_{x \in Y} U_Y(x)$  and all  $A \in p$  the assumption that  $\{t \in S : t^{-1}sA \in p\}$  is syndetic for every  $s \in S$ , implies that  $\{t \in S : t^{-1}A \in p\} \neq \emptyset$ . Then  $K(\beta S) = \bigcap_{x \in Y} U_Y(x)$ .*



*Proof.* Assume that  $p \in \bigcap_{x \in Y} U_Y(x) \setminus K(\beta S)$ . By Theorem 3.4(2),  $\beta Sp \subseteq K(\beta S)$  so  $p \notin \beta Sp$ . Pick  $A \in p$  such that  $\overline{A} \cap \beta Sp = \emptyset$ . Thus  $\{t \in S : t^{-1}A \in p\} = \emptyset$ . We claim that for all  $s \in S$ ,  $\{t \in S : t^{-1}sA\}$  is syndetic. So let  $s \in S$ . By [4, Theorem 4.48] it suffices to let  $L$  be a minimal left ideal of  $\beta S$  and show that there is some  $q \in L$  such that  $\{t \in S : t^{-1}sA \in p\} \in q$ . By Theorem 3.4(2),  $sp \in Lp$  so pick  $q \in L$  such that  $sp = qp$ . Then  $sA \in qp$  so  $\{t \in S : t^{-1}sA \in p\} \in q$  as required.  $\square$

Note that by Theorem 3.4(3),  $K(\beta\mathbb{N}, +) = \bigcap_{x \in Y} U_Y(x)$  while  $1 \notin \beta\mathbb{N} + 1$  so by Theorem 4.2, it is not the case that for all  $x \in X$ ,  $U_X(x) = U_Y(x|_S)$ . On the other hand, given  $p \in K(\beta\mathbb{N}, +)$  one has  $p = q + p$  for some  $p \in K(\beta\mathbb{N}, +)$  so automatically for any  $A \in p$ ,  $\{t \in \mathbb{N} : -t + A \in p\} \neq \emptyset$  so the hypotheses of Theorem 4.8 are valid.

## 5 Recurrence and surjectivity of $T_p$

So far in this paper we have been considering the notion of uniform recurrence. We now introduce a notion which is usually weaker.

**Definition 5.1.** Let  $(X, \langle T_s \rangle_{s \in S})$  be a dynamical system. The point  $x \in X$  is *recurrent* if and only if for each neighborhood  $V$  of  $x$  in  $X$ ,  $\{s \in S : T_s(x) \in V\}$  is infinite.

If all syndetic subsets of a semigroup  $S$  are infinite, then any uniformly recurrent point of  $X$  is recurrent. This is not always the case. For example, if  $S$  is a left zero semigroup and  $x \in S$ , then  $x$  is uniformly recurrent in the dynamical system  $(\beta S, \langle \lambda_s \rangle_{s \in S})$  but is not recurrent. (We have that  $\{x\}$  is a neighborhood of  $x$  and  $\{s \in S : \lambda_s(x) \in \{x\}\} = \{x\}$ , which is syndetic, but finite.)

The following characterization of recurrence is very similar to the characterization of uniform recurrence in [4, Theorem 19.23]. Part of the results depend on the assumption that  $S^*$  is a subsemigroup of  $\beta S$ . There is a characterization of  $S^*$  as a subsemigroup in [4, Theorem 4.28]. By [4, Corollary 4.29 and Theorem 4.31] it is sufficient that  $S$  be right cancellative or weakly left cancellative.

**Theorem 5.2.** Let  $(X, \langle T_s \rangle_{s \in S})$  be a dynamical system. Statements (a) and (b) are equivalent and imply statements (c) and (d), which are equivalent. If  $S^*$  is a subsemigroup of  $\beta S$ , then all four statements are equivalent.

- (a) There exists an idempotent  $p \in S^*$  such that  $T_p(x) = x$ .
- (b) There exist  $y \in X$  and an idempotent  $p \in S^*$  such that  $T_p(y) = x$ .
- (c) There exists  $p \in S^*$  such that  $T_p(x) = x$ .
- (d)  $x$  is recurrent.

*Proof.* Trivially (a) implies (b) and (a) implies (c). To see that (b) implies (a), pick  $y \in X$  and an idempotent  $p \in S^*$  such that  $T_p(y) = x$ . Then  $x = T_p(y) = T_{pp}(y) = T_p(T_p(y)) = T_p(x)$ .

To see that (c) implies (d), pick  $p \in S^*$  such that  $T_p(x) = x$ . Let  $V$  be a neighborhood of  $x$ . Then  $\{s \in S : T_s(x) \in V\} \in p$  so  $\{s \in S : T_s(x) \in V\}$  is infinite.

To see that (d) implies (c), assume that  $x$  is recurrent and for each neighborhood  $V$  of  $x$ , let  $D_V = \{s \in S : T_s(x) \in V\}$ . Then any finite subfamily of  $\{D_V : V \text{ is a neighborhood of } x\}$  has infinite intersection so pick by [4, Corollary 3.14] some  $p \in S^*$  such that  $\{D_V : V \text{ is a neighborhood of } x\} \subseteq p$ . Then  $T_p(x) = x$ .

Now assume that  $S^*$  is a semigroup. To see that (c) implies (a), pick  $p \in S^*$  such that  $T_p(x) = x$  and let  $E = \{q \in S^* : T_q(x) = x\}$ . Since  $S^*$  is a subsemigroup of  $\beta S$ , we have that  $E$  is a subsemigroup of  $\beta S$ . We claim that  $E$  is closed. To see this, let  $q \in \beta S \setminus E$ . If  $q \in S$ , then  $\{q\}$  is a neighborhood of  $q$  missing  $E$ , so assume that  $q \in S^*$ . Pick an open neighborhood  $U$  of  $T_q(x)$  such that  $x \notin \text{cl}U$  and let  $A = \{s \in S : T_s(x) \in U\}$ . Then  $\overline{A}$  is a neighborhood of  $q$  which misses  $E$ . Since  $E$  is a compact right topological semigroup, there is an idempotent in  $E$ .  $\square$

Recall that in any dynamical system,  $(X, \langle T_s \rangle_{s \in S})$ ,  $K(\beta S) \subseteq \bigcap_{x \in X} U_X(x)$  and we have obtained sufficient conditions for equality.

**Theorem 5.3.** *Let  $(X, \langle T_s \rangle_{s \in S})$  be a dynamical system, let  $p \in \beta S$ , and assume that  $T_p : X \rightarrow X$  is surjective and  $K(\beta S) = \bigcap_{x \in X} U_X(x)$ . Then for any  $q \in \beta S$ ,  $qp \in K(\beta S)$  if and only if  $q \in K(\beta S)$ .*

*Proof.* Let  $q \in \beta S$ . The sufficiency is trivial, so assume that  $qp \in K(\beta S)$ . It suffices to show that  $q \in \bigcap_{x \in X} U(x)$ , so let  $x \in X$  be given. Pick  $y \in X$  such that  $T_p(y) = x$ . Then  $T_q(x) = T_q(T_p(y)) = T_{qp}(y)$ . Since  $qp \in U(y)$  we have  $T_{qp}(y)$  is uniformly recurrent, and so  $T_q(x) \in U(x)$  as required.  $\square$

**Definition 5.4.** Let  $(X, \langle T_s \rangle_{s \in S})$  be a dynamical system. Then  $NS = NS_X = \{p \in \beta S : T_p \text{ is not surjective}\}$ .

We have seen that  $U(x)$  is always a left ideal of  $\beta S$ .

**Lemma 5.5.** *Let  $(X, \langle T_s \rangle_{s \in S})$  be a dynamical system. If  $NS \neq \emptyset$ , then  $NS$  is a right ideal of  $\beta S$ .*

*Proof.* Given  $p \in NS$  and  $q \in \beta S$ , the range of  $T_{pq}$  is contained in the range of  $T_p$ .  $\square$

**Lemma 5.6.** *Let  $(X, \langle T_s \rangle_{s \in S})$  be a dynamical system. If there is some  $x \in X$  such that  $x$  is not recurrent, then  $\{p \in S^* : pp = p\} \subseteq NS$ .*

*Proof.* Pick  $x \in X$  such that  $x$  is not recurrent and let  $p$  be an idempotent in  $S^*$ . We claim that  $x$  is not in the range of  $T_p$ , so suppose instead we have  $y \in X$  such that  $T_p(y) = x$ . Then by Theorem 5.2,  $x$  is recurrent.  $\square$

We shall establish a strong connection between the surjectivity of  $T_p$  and  $p$  being right cancelable in  $\beta S$ . The purely algebraic result in Theorem 5.8 will be useful.

**Lemma 5.7.** *Let  $S$  be a countable right cancellative and weakly left cancellative semigroup and let  $B$  be an infinite subset of  $S$ . There is an infinite subset  $D$  of  $B$  with the property that whenever  $s$  and  $t$  are distinct members of  $S$ , there is a finite subset  $F$  of  $D$  such that  $sa \neq tb$  whenever  $a, b \in D \setminus F$ .*

*Proof.* Let  $\Delta = \{(s, s) : s \in S\}$  and enumerate  $(S \times S) \setminus \Delta$  as  $\langle (s_n, t_n) \rangle_{n=1}^\infty$ . Pick  $a_1 \in B$ . Assume  $n \in \mathbb{N}$  and we have chosen  $\langle a_i \rangle_{i=1}^n$ . Let  $W_n = \{b \in S : \text{there exist } i, j \in \{1, 2, \dots, n\} \text{ such that } s_i a_j = t_i b \text{ or } s_i b = t_i a_j\}$ . Then  $W_n$  is the union of finitely many left solution sets, so is finite. Pick  $a_{n+1} \in B \setminus (W_n \cup \{a_1, a_2, \dots, a_n\})$ .

Let  $D = \{a_n : n \in \mathbb{N}\}$ . Let  $s$  and  $t$  be distinct members of  $S$  and pick  $n$  such that  $(s, t) = (s_n, t_n)$ . Let  $F = \{a_i : i \in \{1, 2, \dots, n\}\}$ . To see that  $F$  is as required, let  $a, b \in D \setminus F$  and suppose  $sa = tb$ . Then by right cancellation,  $a \neq b$ . Pick  $m > n$  and  $r > n$  such that  $a = a_m$  and  $b = a_r$ . If  $m < r$ , then  $a_r \in W_{r-1}$ . If  $r < m$ , then  $a_m \in W_{m-1}$ .  $\square$

**Theorem 5.8.** *Let  $S$  be a countable cancellative semigroup. If  $p \in \beta S \setminus K(\beta S)$ , then there exists an infinite  $D \subseteq S$  such that for every  $r \in D^*$ ,  $rp$  is right cancelable in  $\beta S$ .*

*Proof.* Choose  $q \in K(\beta S)$ . We first claim that for each  $s \in S$ ,  $sp \notin K(\beta S)$  and in particular,  $sp \notin \beta Sqp$ . So suppose we have  $sp \in K(\beta S)$ . Then  $sp$  is in a minimal left ideal  $L$  of  $\beta S$ . Pick an idempotent  $r \in L$ . By [4, Lemma 1.30],  $sp = spr$ . By [4, Lemma 8.1]  $s$  is left cancelable in  $\beta S$  so  $p = pr$ , and thus  $p \in K(\beta S)$ . This contradiction establishes the claim. For each  $s \in S$ , pick  $U_s \in sp$  such that  $\overline{U_s} \cap \beta Sqp = \emptyset$ . For each  $s, t \in S$ , there exists  $V_{s,t} \in q$  such that  $\overline{U_s} \cap t\overline{V_{s,t}}p = \emptyset$  because  $\lambda_t \circ \rho_p(q) \in \beta S \setminus \overline{U_s}$ .

By [4, Theorem 3.36], there exists an infinite subset  $B$  of  $S$  such that  $B^* \subseteq \bigcap_{s,t \in S} \overline{V_{s,t}}$ . Then for every  $r \in B^*$  and every  $s, t \in S$ ,  $trp \notin \overline{U_s}$ .

By Lemma 5.7 pick an infinite subset  $D$  of  $B$  such that, whenever  $s$  and  $t$  are distinct elements of  $S$ , there is a finite subset  $F$  of  $D$  such that  $sa \neq tb$  whenever  $a, b \in D \setminus F$ . Enumerate  $D$  as  $\langle d_n \rangle_{n=1}^\infty$  and for each distinct  $s$  and  $t$  in  $S$ , pick  $n_{s,t} \in \mathbb{N}$  such that  $sd_m \neq td_n$  whenever  $m, n > n_{s,t}$ .

We claim that, for every  $r \in D^*$ ,  $rp$  is right cancelable in  $\beta S$ . We shall apply [4, Theorem 3.40] three times.

Assume that  $q_1rp = q_2rp$ , where  $q_1$  and  $q_2$  are distinct elements of  $\beta S$ . Let  $A_1$  and  $A_2$  be disjoint subsets of  $S$  which are members of  $q_1$  and  $q_2$  respectively. Since  $q_1rp \in cl(A_1rp)$  and  $q_2rp \in cl(A_2rp)$ , an application of [4, Theorem 3.40] shows that either  $A_1rp \cap cl(A_2rp) \neq \emptyset$  or  $A_2rp \cap cl(A_1rp) \neq \emptyset$ , and without loss of generality, we may assume that the former holds. Thus we have some  $s \in A_1$  and  $q' \in \overline{A_2}$  such that  $srp = q'rp$ . Now  $srp \in cl(sDp)$  and  $q'rp \in cl((S \setminus \{s\})rp)$ , so either  $sDp \cap cl((S \setminus \{s\})rp) \neq \emptyset$  or  $(S \setminus \{s\})rp \cap cl(sDp) \neq \emptyset$ . We thus have either

- (i)  $sDp \cap cl((S \setminus \{s\})rp) \neq \emptyset$ , in which case we choose  $d \in D$  and  $y \in \beta S$  such that  $sdp = yrp$ ; or
- (ii)  $sDp \cap cl((S \setminus \{s\})rp) = \emptyset$ , in which case we pick  $t \in S \setminus \{s\}$  and  $r' \in \overline{D}$  such that  $sr'p = trp$ . Since  $sDp \cap cl((S \setminus \{s\})rp) = \emptyset$ , we have  $r' \in D^*$ .

Suppose that (i) holds. Then  $U_{sd} \in sdp$  so  $\{v \in S : v^{-1}U_{sd} \in rp\} \in y$ , so pick  $v \in S$  such that  $U_{sd} \in vrp$ . But  $r \in V_{sd,v}$ , so this is a contradiction. Thus (ii) holds.

Now  $sr'p \in cl\{sd_m p : m > n_{s,t}\}$  and  $trp \in cl\{td_m p : m > n_{s,t}\}$  so, essentially without loss of generality, we have  $\{sd_m p : m > n_{s,t}\} \cap cl\{td_m p : m > n_{s,t}\} \neq \emptyset$ . (We have distinguished between  $s$  and  $t$  at this stage, but the arguments below with  $s$  and  $t$  interchanged remain valid.) Thus either

- (iii) there exist  $m, n > n_{s,t}$  such that  $sd_m p = td_n p$ ; or
- (iv) there exist  $m > n_{s,t}$  and  $r'' \in D^*$  such that  $sd_m p = tr''p$ .

If (iii) holds, then by [4, Lemma 6.28],  $sd_m = td_n$ , contradicting the choice of  $n_{s,t}$ . So (iv) holds. But  $r'' \in V_{sd_m,t}$  so  $tr''p \notin U_{sd_m}$ , a contradiction.  $\square$

We now present several results about the dynamical systems considered in Section 3.

**Lemma 5.9.** *Let  $S$  be a semigroup and let  $p$  be a right cancelable element of  $\beta S$ . Then for any clopen subset  $E$  of  $\beta Sp$ , there is some  $A \subseteq S$  such that  $E = \overline{A} \cap \beta Sp$ .*

*Proof.* Let  $E$  be a clopen subset of  $\beta Sp$ . Let  $\mathcal{D} = \{\overline{D} \cap \beta Sp : D \subseteq S \text{ and } \overline{D} \cap \beta Sp \subseteq E\}$ . Since  $\{\overline{D} \cap \beta Sp : D \subseteq S\}$  is a basis for the topology of  $\beta Sp$  and  $E$  is open in  $\beta Sp$ , we have that  $E = \bigcup \mathcal{D}$ . Since  $E$  is compact, pick finite  $\mathcal{F} \subseteq \mathcal{D}$  such that  $E = \bigcup_{D \in \mathcal{F}} (\overline{D} \cap \beta Sp)$  and let  $A = \bigcup \mathcal{F}$ .  $\square$

**Theorem 5.10.** *Let  $S$  be a semigroup. Let  $(Y, \langle T_s \rangle_{s \in S})$  be the dynamical system of Lemma 1.2 determined by  $R = S$ . Let  $p \in \beta S$ . If  $p$  is right cancelable in  $\beta S$ , then  $T_p : Y \rightarrow Y$  is surjective.*

*Proof.* Note that since  $\rho_p : \beta S \rightarrow \beta Sp$  is injective and takes closed sets to closed sets, it is a homeomorphism.

To see that  $T_p$  is surjective, let  $z \in Y$ , let  $B = \{s \in S : z(s) = 1\}$ , and let  $E = \rho_p[\overline{B}]$ . Then  $E$  is clopen in  $\beta Sp$  so by Lemma 5.9 pick  $A \subseteq S$  such that  $E = \overline{A} \cap \beta Sp$ . Let  $x$  be the characteristic function of  $A$  in  $Y$ . We claim that  $T_p(x) = z$ . For this, it suffices that for each  $s \in S$ ,  $\{t \in S : T_t(x)(s) = z(s)\} \in p$ . So let  $s \in S$ . Note that  $\{t \in S : T_t(x)(s) = 1\} = \{t \in S : x(st) = 1\} = s^{-1}A$ . Also  $s^{-1}A \in p$  if and only if  $s \in \rho_p^{-1}[\overline{A} \cap \beta Sp]$  so  $s \in B$  if and only if  $s^{-1}A \in p$ .

If  $z(s) = 1$ , then  $s \in B$  so  $s^{-1}A \in p$  so  $\{t \in S : T_t(x)(s) = z(s)\} \in p$ . If  $z(s) = 0$ , then  $s \notin B$  so  $s^{-1}A \notin p$  so  $\{t \in S : T_t(x)(s) = z(s)\} \in p$ .  $\square$

Notice that the hypotheses of the following corollary hold if  $S$  has any right cancelable element.

**Corollary 5.11.** *Let  $S$  be a semigroup. Let  $(Y, \langle T_s \rangle_{s \in S})$  be the dynamical system of Lemma 1.2 determined by  $R = S$ . Let  $p \in \beta S$ . Assume that for whenever  $q$  and  $r$  are distinct elements of  $\beta S$ , there exists  $s \in S$  such that  $sq \neq sr$ . Then  $T_p : Y \rightarrow Y$  is surjective if and only if  $p$  is right cancelable in  $\beta S$ .*

*Proof.* The necessity is Theorem 5.10.

So assume that  $T_p$  is surjective and suppose that we have distinct  $q$  and  $r$  in  $\beta S$  such that  $qp = rp$ . We claim that  $T_q = T_r$ . To see this, let  $x \in Y$  be given. Pick  $z \in Y$  such that  $T_p(z) = x$ . Then  $T_q(x) = T_q(T_p(z)) = T_{qp}(z) = T_{rp}(z) = T_r(T_p(z)) = T_r(x)$ .

Pick  $s \in S$  such that  $sq \neq sr$ , pick  $A \in sq \setminus sr$ , and let  $x$  be the characteristic function of  $A$  in  $Y$ . Then  $A \subseteq \{t \in S : T_t(x)(s) = 1\}$  so  $T_q(x)(s) = 1$  and  $S \setminus A \subseteq \{t \in S : T_t(x)(s) = 0\}$  so  $T_r(x)(s) = 1$ .  $\square$

**Theorem 5.12.** *Let  $S$  be a semigroup and let  $Q = S \cup \{e\}$  where  $e$  is an identity adjoined to  $S$ . Let  $(X, \langle T_s \rangle_{s \in S})$  be the dynamical system of Lemma 1.2 determined by  $R = Q$  and let  $p \in \beta S$ . Then  $T_p : X \rightarrow X$  is surjective if and only if  $p$  is right cancelable in  $\beta Q$ .*

*Proof.* Sufficiency. Note that  $\rho_p : \beta S \rightarrow \beta Sp$  is a homeomorphism. Note also that  $p \notin \beta Sp$ . (If we had  $p = qp$  for some  $q \in \beta S$ , then we would have  $ep = qp$ .) Let  $x \in X$  and let  $B = \{s \in S : x(s) = 1\}$ . By Lemma 5.9, pick  $A \subseteq S$  such that  $\rho_p[\overline{B}] = \overline{A} \cap \beta Sp$ . Pick  $P \in p$  such that  $\overline{P} \cap \beta Sp = \emptyset$ . If  $x(e) = 1$ , let  $D = A \setminus P$ . If  $x(e) = 0$ , let  $D = A \cup B$ . Let  $z$  be the characteristic function of  $D$  in  $X$ .

We claim that  $T_p(z) = x$ . As in the proof of Theorem 5.10, we see that for  $s \in S$ ,  $T_p(z)(s) = x(s)$ . Regardless of the value of  $x(e)$ , we have that  $P \subseteq \{s \in S : T_s(z)(e) = x(e)\}$ , so  $T_p(z)(e) = x(e)$ .

Necessity. Suppose that  $T_p$  is surjective and we have  $q \neq r$  in  $\beta Q$  such that  $qp = rp$ . Assume first that  $e \in \{q, r\}$ , so without loss of generality,  $q = e$ . Let

$x$  be the characteristic function of  $S$  in  $X$  and pick  $z \in X$  such that  $T_p(z) = x$ . Then  $0 = x(e) = T_p(z)(e) = T_{rp}(z)(e) = T_r(T_p(z))(e) = T_r(x)(e) = 1$ , a contradiction.

So we can assume that  $q$  and  $r$  are in  $\beta S$ . Pick  $A \in q \setminus r$  and let  $A$  be the characteristic function of  $A$  in  $X$ . Pick  $z \in X$  such that  $T_p(z) = x$ . Then  $0 = T_r(x)(e) = T_{rp}(z)(e) = T_{qp}(z)(e) = T_q(T_p(z))(e) = T_q(x)(e) = 1$ , a contradiction.  $\square$

**Theorem 5.13.** *Let  $S$  be a countable semigroup which can be embedded in a group and assume that  $S$  can be enumerated as  $\langle s_t \rangle_{t=0}^\infty$  so that if  $u, v \in S$ ,  $i, j \in \omega$  with  $i < j$ , and  $s_i u = s_j v$ , then  $s_0 s_i^{-1} s_j \in S$ . Let  $(Y, \langle T_s \rangle_{s \in S})$  be the dynamical system of Lemma 1.2 determined by  $R = S$  and let  $p \in \beta S$ . The  $T_p$  is surjective if and only if there exists  $x \in Y$  such that  $T_p(x)$  is the characteristic function of  $\{s_0\}$  in  $Y$ .*

*Proof.* The necessity is trivial. Assume that we have  $x \in Y$  such that  $T_p(x)$  is the characteristic function of  $\{s_0\}$  in  $Y$ . For  $m \in \mathbb{N}$ , let  $D_m = \{s_0 s_i^{-1} s_j : i, j \in \{0, 1, \dots, m\}, i < j, \text{ and } s_0 s_i^{-1} s_j \in S\}$  and note that  $s_0 \notin D_m$ . For each  $m \in \mathbb{N}$ , let

$$B_m = \{s \in S : T_s(x) \in \pi_{s_0}^{-1}[\{1\}] \cap \bigcap_{i=1}^m \pi_{s_i}^{-1}[\{0\}] \cap \bigcap_{r \in D_m} \pi_r^{-1}[\{0\}]\},$$

and note that  $B_m \in p$ . We claim that if  $m, k \in \mathbb{N}$ ,  $u \in B_m$ ,  $v \in B_k$ ,  $i \in \{0, 1, \dots, m\}$ ,  $j \in \{0, 1, \dots, k\}$ , and  $s_i u = s_j v$ , then  $i = j$ . Suppose instead we have such  $m, k, u, v, i, j$  with  $i \neq j$  and assume without loss of generality that  $i < j$ . Then  $u = s_i^{-1} s_j v$ . By assumption  $s_0 s_i^{-1} s_j \in S$  so  $s_0 s_i^{-1} s_j \in D_k$ . Since  $u \in B_m$ ,  $1 = T_u(x)(s_0) = x(s_0 u)$ . Since  $v \in B_k$  and  $s_0 s_i^{-1} s_j \in D_k$ ,  $0 = T_v(x)(s_0 s_i^{-1} s_j) = x(s_0 s_i^{-1} s_j v)$ , a contradiction.

Now to see that  $T_p$  is surjective, let  $y \in Y$  be given. Define  $w \in Y$  as follows. If  $m \in \mathbb{N}$ ,  $u \in B_m$ , and  $i \in \{0, 1, \dots, m\}$ , then  $w(s_i u) = y(s_i)$ . For  $s \in S$  which is not of the form  $s_i u$  for some  $m \in \mathbb{N}$ ,  $u \in B_m$ , and  $i \in \{0, 1, \dots, m\}$ , define  $w(s)$  at will. To see that  $T_p(w) = y$ , let  $U$  be a neighborhood of  $y$ . Pick  $m \in \mathbb{N}$  such that  $\bigcap_{i=0}^m \pi_i^{-1}[\{y(s_i)\}] \subseteq U$ . Then  $B_m \subseteq U$ .  $\square$

The following is an immediate corollary of Theorem 5.13.

**Corollary 5.14.** *Let  $S$  be a countable group with identity  $e$ , let  $(Y, \langle T_s \rangle_{s \in S})$  be the dynamical system of Lemma 1.2 determined by  $R = S$ , and let  $p \in \beta S$ . The following statements are equivalent.*

- (a)  $T_p$  is surjective.
- (b) For each  $s \in S$ , there exists  $x \in Y$  such that  $T_p(x)$  is the characteristic function of  $\{s\}$ .
- (c) There exists  $x \in Y$  such that  $T_p(x)$  is the characteristic function of  $\{e\}$ .

Notice that the hypotheses of the following theorem hold if  $S$  is very weakly left cancellative and right cancellative. If  $\kappa$  is regular, the assumption that for any subset  $D$  of  $S$  with fewer than  $\kappa$  members,  $|\{e \in S : (\exists s \in D)(\exists t \in D \setminus \{s\})(se = te)\}| < \kappa$  can be replaced by the assumption that for all distinct  $s$  and  $t$  in  $S$ ,  $|\{e \in S : se = te\}| < \kappa$ .

**Theorem 5.15.** *Let  $S$  be a semigroup with  $|S| = \kappa \geq \omega$  which is very weakly left cancellative and has the property that for any subset  $D$  of  $S$  with fewer than  $\kappa$  members,  $|\{e \in S : (\exists s \in D)(\exists t \in D \setminus \{s\})(se = te)\}| < \kappa$ . Let  $(Y, \langle T_s \rangle_{s \in S})$  be the dynamical system of Lemma 1.2 determined by  $R = S$ . There is a dense open subset  $W$  of  $U_\kappa(S)$  such that for every  $p \in W$ ,  $p$  is right cancelable in  $\beta S$  and  $T_p : Y \rightarrow Y$  is surjective.*

*Proof.* We show that for any  $C \in [S]^\kappa$ , there exists  $B \in [C]^\kappa$  such that for every  $p \in \overline{B} \cap U_\kappa(S)$ ,  $p$  is right cancelable in  $\beta S$  and  $T_p : Y \rightarrow Y$  is surjective.

Enumerate  $S$  as  $\langle s_\gamma \rangle_{\gamma < \kappa}$ . Choose  $t_0 \in C$ . Let  $0 < \alpha < \kappa$  and assume that we have chosen  $\langle t_\delta \rangle_{\delta < \alpha}$  in  $C$  satisfying the following inductive hypotheses:

- (1) If  $\gamma < \delta$ , then  $t_\gamma \neq t_\delta$ .
- (2) If  $\gamma < \delta$ ,  $\mu < \beta \leq \delta$ , and  $\mu \neq \gamma$ , then  $s_\gamma t_\delta \neq s_\mu t_\beta$ .

The hypotheses are satisfied for  $\delta = 0$ . Let  $E = \{e \in S : (\exists \mu < \beta \leq \alpha)(s_\mu e = s_\beta e)\}$ . For  $\mu < \beta < \alpha$  and  $\gamma < \alpha$  let  $A_{\gamma, \mu, \beta} = \{t \in S : s_\gamma t = s_\mu t_\beta\}$ . Then each  $A_{\gamma, \mu, \beta}$  is a left solution set. Pick

$$t_\alpha \in C \setminus (\{t_\gamma : \gamma < \alpha\} \cup E \cup \{\bigcup_{\gamma < \alpha} \bigcup_{\beta < \alpha} \bigcup_{\mu < \beta} A_{\gamma, \mu, \beta}\}).$$

Hypothesis (1) is trivially satisfied and if  $\mu < \beta < \alpha$  and  $\gamma < \alpha$ , then  $t_\alpha \notin A_{\gamma, \mu, \beta}$  so  $s_\gamma t_\alpha \neq s_\mu t_\beta$ . If  $\mu < \beta = \alpha$  and  $\gamma < \alpha$ , then  $t_\alpha \notin E$  so  $s_\gamma t_\alpha \neq s_\mu t_\beta$ .

Let  $B = \{t_\alpha : \alpha < \kappa\}$  and let  $p \in \overline{B} \cap U_\kappa(S)$ . To see that  $p$  is right cancelable in  $\beta S$ , let  $q \neq r \in \beta S$  and suppose that  $qp = rp$ . Pick subsets  $C$  and  $D$  of  $S$  such that  $C \cap D = \emptyset$  and  $C \in q$  and  $D \in r$ . Then  $H = \{s_\gamma t_\alpha : \gamma < \alpha \text{ and } s_\gamma \in C\} \in qp$ . (To see this, let  $s_\gamma \in C$ . Then  $\{t_\alpha : \gamma < \alpha < \kappa\} \subseteq s_\gamma^{-1}H$ .) Similarly,  $\{s_\mu t_\beta : \mu < \beta \text{ and } s_\mu \in D\} \in rp$ . Since these sets are disjoint by hypothesis (2), we have a contradiction.

The fact that  $T_p$  is surjective follows from Theorem 5.10.  $\square$

**Lemma 5.16.** *Let  $S$  be a cancellative semigroup, let  $a \in S$ , and let  $(Y, \langle T_s \rangle_{s \in S})$  be the dynamical system of Lemma 1.2 determined by  $R = S$ . If  $x$  is the characteristic function of  $\{a\}$  in  $Y$ , then  $x$  is not a recurrent point.*

*Proof.* We claim that  $|\{s \in S : T_s(x)(a) = 1\}| \leq 1$ . Indeed, if  $x(as) = 1$ , then  $as = a$  so by left cancellation,  $s$  is a left identity for  $S$  and then by right cancellation,  $s$  is a two sided identity for  $S$ .  $\square$

We have seen that  $U(x)$  is always a left ideal of  $\beta S$  and that  $NS$  is a right ideal of  $\beta S$  provided it is nonempty.

**Theorem 5.17.** *Let  $S$  be a countable cancellative semigroup. Let  $(Y, \langle T_s \rangle_{s \in S})$  be the dynamical system of Lemma 1.2 determined by  $R = S$ . Then  $NS_Y$  is not a left ideal of  $\beta S$ .*

*Proof.* By [4, Corollary 6.33] pick an idempotent  $p \in \beta S \setminus K(\beta S)$ . By Theorem 5.8 pick  $r \in \beta S$  such that  $rp$  is right cancelable in  $\beta S$ . By Lemma 5.16 and Theorem 5.6,  $p \in NS$  and by Theorem 5.10,  $rp \notin NS$ .  $\square$

If  $S$  is commutative, then by [4, Exercise 4.4.9] and Theorem 5.5, if  $NS \neq \emptyset$ , then  $clNS$  is a two sided ideal of  $\beta S$ . The following theorem shows that this may fail if  $S$  is not commutative.

**Theorem 5.18.** *Let  $S$  be the free semigroup on the alphabet  $\{a, b\}$  (where  $a \neq b$ ). Let  $(Y, \langle T_s \rangle_{s \in S})$  be the dynamical system of Lemma 1.2 determined by  $R = S$ . Then  $NS \neq \emptyset$  and  $clNS$  is not a left ideal of  $\beta S$ .*

*Proof.* Let  $p$  be an idempotent in  $\beta S$  with  $\{a^n : n \in \mathbb{N}\} \in p$ . By Lemma 5.16 and Theorem 5.6,  $p \in NS$ . We will show that  $bp \notin clNS$ . Let  $B = \{ba^n : n \in \mathbb{N}\}$ . Then  $B \in bp$ . We shall show that  $\overline{B} \cap NS = \emptyset$ . So let  $q \in \overline{B}$ . Let  $s_0 = a$  and let  $\langle s_n \rangle_{n=1}^\infty$  enumerate  $S \setminus \{a\}$  so that if the length of  $s_i$  is less than the length of  $s_j$ , then  $i < j$ . By Theorem 5.13, to see that  $T_q$  is surjective, it suffices to show that there is some  $x \in Y$  such that  $T_q(x)$  is the characteristic function of  $\{a\}$ .

Let  $x$  be the characteristic function of  $\{aba^n : n \in \mathbb{N}\}$  in  $Y$ . Let  $U$  be a neighborhood of  $\chi_{\{a\}}$  and pick  $F \in \mathcal{P}_f(S \setminus \{a\})$  such that  $\pi_a^{-1}[\{1\}] \cap \bigcap_{y \in F} \pi_y^{-1}[\{0\}] \subseteq U$ . It suffices to show that  $B \subseteq \{w \in S : T_w(x) \in \pi_a^{-1}[\{1\}] \cap \bigcap_{y \in F} \pi_y^{-1}[\{0\}]\}$ . So let  $ba^n \in B$ . Then  $T_{ba^n}(x)(a) = x(aba^n) = 1$  and for  $y \in F$ ,  $T_{ba^n}(x)(y) = x(yba^n) = 0$ .  $\square$

We remark that Theorem 5.18 remains valid if  $S$  is the free semigroup on a countably infinite alphabet.

## References

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